A Generalized Model for Multidimensional Intransitivity

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Objectives

- Predict pairwise matchup.
- Learn multidimensional reprensentations from pairwise comparison data.

Intransitivity

An intuitive example a rock-paper-scissors game, the pairwise matchup result is judged by three rules: $\{o_{paper} \succ o_{rock}, o_{rock} \succ o_{scissors}$ and $o_{scissors} \succ o_{paper}\}$. A transitive model results in a transitive dominance $o_{paper} \succ o_{scissors}$, that violates the third rule $o_{scissors} \succ o_{paper}$.

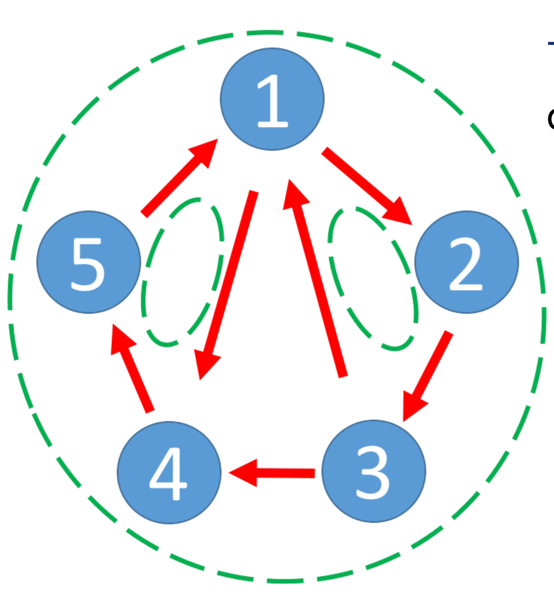

Figure: A DAG illustration of the observed game in Table 1

Table: A toy model that demosntates the subtle accuracy deterioration in terms of test accuracy

winner	loser	#win	#lose	GT	pred$_{trans}$	pred$_{intrans}$
1	2	10	5	✓	✓	✓
1	3	1	2	✓	x	✓
1	4	10	5	✓	✓	✓
1	5	1	2	✓	x	✓
2	3	10	5	✓	✓	✓
3	4	10	5	✓	✓	✓
3	5	10	5	✓	✓	✓
4	5	10	5	✓	✓	✓
Test Accuracy					0.6458	0.6667

Related works

Problem setting

- Candidate players a_i and $b_i \in \mathbf{P}$ with $|\mathbf{P}| = M$.
- Pairwise matchup records $x_i(a_i, b_i) \in \{0,1\}$, $i = [1{:}N]$
- $o_a \succ o_b := (a, b, 1, 0)$, and 4-tuples in $x_{aggregate}(a, b) = (a, b, n_a, n_b)$

1. Bradley-Terry Model (Bradley et al., 1952)

$$P(o_a \succ o_b) = \frac{\exp(\gamma_a)}{\exp(\gamma_a) + \exp(\gamma_b)} = \frac{1}{1 + \exp(-M_{ab})} \quad (1)$$

γ_p is the ability of winning for player p.

$$M_{ab} = \gamma_a - \gamma_b$$

2. Blade-Chest Model (Chen et al., 2016a, 2016b)

$$M^{BCI}(a, b) = \mathbf{a}_{blade}^T \cdot \mathbf{b}_{chest} - \mathbf{b}_{blade}^T \cdot \mathbf{a}_{chest} \quad (2)$$

Proposed method

- A generalized model with more expressiveness.
- A quantitative evaluation of the existence of intransitivity in datasets.

Matchup function

$$M^G(a, b) = \mathbf{a}^T \Sigma \mathbf{b} + \mathbf{a}^T \Gamma \mathbf{a} - \mathbf{b}^T \Gamma \mathbf{b} \quad (3)$$

where $\mathbf{a}$ and $\mathbf{b} \in \mathbb{R}^d$ are the d-dimensional representation for player a and player b and $\Sigma, \Gamma \in \mathbb{R}^{d \times d}$ are the *transitive matrices*.

Symmetry $Pr(a \prec b) = 1 - Pr(a \succ b)$ requires

$$\Sigma = -\Sigma^T \quad (4)$$

the resulted 'hard' constrained optimization can be resolved by

$$\Sigma = \Sigma' - \Sigma'^T \quad (5)$$

with $\Sigma' \in \mathbb{R}^d$

Training Maximum likelihood by SGD.

Regularization $R_1(D|\theta) = \Sigma_{a \in \mathbf{P}} \frac{1}{2} \|\mathbf{a}\|_2^2$, $R_2(D|\theta) = \|\Sigma'\|_F$ and $R_3(D|\theta) = \|\Gamma\|_F$

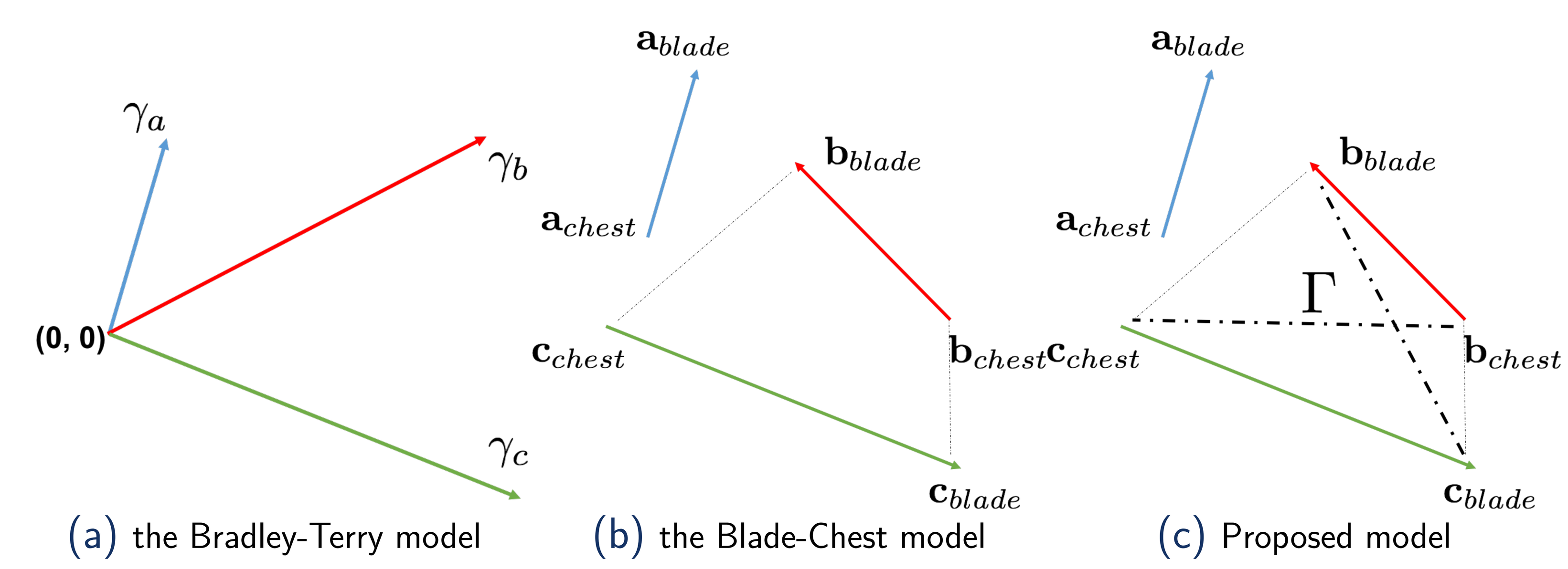

Figure: An illustration of the three models

Expressiveness

The proposed method breaks the numerical conjuction that appeared in Blade-Chest model. Suppose that we have blade and chest vectors for player a, $\mathbf{a}_{blade}$ and $\mathbf{a}_{chest} \in \mathbb{R}^{d'}$, where $d' = 3$, then integrate them into a generallized vector $\mathbf{a}_{general} \in \mathbb{R}^{2d'}$ defined by

$$\mathbf{a}_{general} = \begin{bmatrix} \mathbf{a}_{blade} \\ \mathbf{a}_{chest} \end{bmatrix} \quad (6)$$

Theorem [Expressiveness] Given the proposed matchup formulation in $2d'$-dimensional space, the proposed model degenerates to a Blade-Chest-Inner (BCI) model in d'-dimensional space, under mild condition

$$\|\mathbf{a}\|_2^2 = \|\mathbf{b}\|_2^2 \quad (7)$$

$$\|\Gamma'\|_F \to 0$$

and,

$$\Sigma = \begin{bmatrix} 0 & I_{d' \times d'} \\ -I_{d' \times d'} & 0 \end{bmatrix}$$

Datasets and Experiments

Question 1: How many intransitive relationships are there inside the datasets? Answer:

Table: A Summary of the real world datasets

DATASET	# of Players	# of Records	isIntrans	Intrans@3	#PlayerIntrans@3
SushiA	10	100000	x	0.00%	0/10
SushiB	100	25000	✓	26.87%	92/100
Jester	100	891404	✓	1.77%	97/100
MovieLens100K	1682	139982	✓	0.19%	1130/1682
ElectionA$_5$	16	44298	✓	0.44 %	6/16
SF4$_{5000}$	35	5000	✓	23.86%	34/35
Dota	757	10442	✓	97.58%	550/757

- **isIntrans**: whether the dataset contains intransitivity (x/✓)
- **Intrans@3**: the amount of the rock-paper-scissors-like relationship (%)
- **#PlayerIntrans@3**: the fraction of players invovled in rock-paper-scissors

Question 2: How does the proposed method perform? Answer:

Table: Test accuracy on real-world datasets

DATASET	Naive	Bradley-Terry	Blade-Chest	Our Proposed
SushiA	0.6549 ± 0.0044	0.6549 ± 0.0021	0.6551 ± 0.0038	$\mathbf{0.6551 \pm 0.0027}$
SushiB	0.6466 ± 0.0042	0.6582 ± 0.0077	0.6591 ± 0.0051	$\mathbf{0.6593 \pm 0.0058}$
Jester	0.6216 ± 0.0006	0.6236 ± 0.0028	0.6242 ± 0.0035	$\mathbf{0.6243 \pm 0.0019}$
ElectionA$_5$	0.6507 ± 0.0031	0.6531 ± 0.0038	0.6533 ± 0.0043	$\mathbf{0.6535 \pm 0.0055}$
SF4$_{5000}$	0.5297 ± 0.0102	0.5329 ± 0.0044	0.5329 ± 0.0062	$\mathbf{0.5355 \pm 0.0080}$

References

Bradley et al., Biometrica, 1952

Chen et al., 2016a

Chen et al., 2016b